Time is just an auxiliary parameter

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Abstract: I explain how time-independent approach and time-dependent approach are equivalent.

The relation between time-dependent approach and the stationary approach is given by **Fourier-Laplace transform**: for $\epsilon > 0$

$$R(\lambda + i\epsilon) = (H - (\lambda + i\epsilon))^{-1} = i \int_0^\infty \exp(it[(\lambda + i\epsilon) - H])dt, \tag{1}$$

and

$$R(\lambda - i\epsilon) = (H - (\lambda - i\epsilon))^{-1} = -i \int_{-\infty}^0 \exp(it[(\lambda - i\epsilon) - H])dt. \tag{2}$$

Or conversely, by Fourier's inversion formula we have

$$\exp(-itH) = (2\pi i)^{-1} \int_{-\infty}^\infty (H - (\lambda + i\epsilon))^{-1} e^{-it(\lambda+i\epsilon)} d\lambda \tag{3}$$

for $t \geq 0$, and

$$\exp(-itH) = -(2\pi i)^{-1} \int_{-\infty}^\infty (H - (\lambda - i\epsilon))^{-1} e^{-it(\lambda-i\epsilon)} d\lambda \tag{4}$$

for $t \leq 0$.

Writing $z = \lambda + i\epsilon$, we have

$$\frac{1}{2\pi i}(R(z) - R(\bar{z})) = (2\pi)^{-1} \int_{-\infty}^\infty e^{it\lambda} \exp(-itH) e^{-\epsilon|t|} dt. \tag{5}$$

In particular, we have by inversion formula

$$\exp(-itH) = \lim_{\epsilon \downarrow 0} \frac{1}{2\pi i} \int_{-\infty}^\infty e^{-it\lambda} (R(\lambda + i\epsilon) - R(\lambda - i\epsilon)) d\lambda. \tag{6}$$

When $\epsilon \downarrow 0$, the right-hand-sides of (1) and (2) reflect the behavior of $\exp(-itH)$ as $t \to \pm\infty$.

Local time is most well-defined in classical sense when $t \to \pm\infty$. Thus said fact means that the local time mostly corresponds to the behavior of the boundary values of the resolvents $R(z)$ and $R(\bar{z})$ as $\epsilon \downarrow 0$.

The investigation of the boundary values of the resolvents is called stationary approach, which corresponds to that the approach considers the stationary internal oscillation inside a local system. On the other hand, the investigation of the asymptotic behavior of $\exp(-itH)$ as $t \to \pm\infty$ is called time-dependent approach.

As we have seen, those two approaches are connected by Laplace transform, and as such they are equivalent.

Scattering operator is defined by the following limit if the limit exists:

$$S = \text{s-}\lim_{t\to\infty} \exp(itH_0)\exp(-2itH)\exp(itH_0). \tag{7}$$

Here $H_0 = -\frac{1}{2}\Delta$ is the unperturbed Hamiltonian of $H = H_0 + V$.

In many cases when V is not so large that $H = H_0 + V$ remains selfadjoint, it is known that S is unitary so that the total probability is conserved throughout the scattering process.

By definition S commutes with H_0 so that the Fourier transform $\hat{S} = \mathcal{F}S\mathcal{F}^{-1}$ is decomposable: for any $\lambda > 0$ there is a unitary operator $\mathcal{S}(\lambda)$ on $L^2(S^{n-1})$ such that for all $\lambda > 0$ and $\omega \in S^{n-1}$,

$$(\hat{S}f)(\sqrt{2\lambda}\omega) = (\mathcal{S}(\lambda)f(\sqrt{2\lambda}\cdot))(\omega) \tag{8}$$

for all $f \in L^2(R^n)$.

This family $\{\mathcal{S}(\lambda)\}_{\lambda>0}$ of unitary operators is called a scattering matrix. It represents a behavior of $\exp(-itH)$ at $t \to \pm\infty$.

On the other hand it is known that

$$\mathcal{S}(\lambda) = I - 2\pi i\mathcal{F}_0(\lambda)V\mathcal{F}_0(\lambda)^* + 2\pi i\mathcal{F}_0(\lambda)VR(\lambda+i0)V\mathcal{F}_0(\lambda)^* \tag{9}$$

when V is short-range, where $\mathcal{F}_0(\lambda)$ is a decomposed Fourier transform:

$$\begin{aligned}\mathcal{F}_0(\lambda)f(\omega) &= (2\lambda)^{(n-2)/4}(\mathcal{F}f)(\sqrt{2\lambda}\omega)\\ &= (2\lambda)^{(n-2)/4}(2\pi)^{-n/2}\int_{R^n} e^{-i\sqrt{2\lambda}\omega x}f(x)dx \in L^2(S^{n-1}).\end{aligned}$$

Thus the behavior of $\exp(-itH)$ as $t \to \pm\infty$ is represented by a boundary value of $R(\lambda + i\epsilon)$ as $\epsilon \downarrow 0$.

The boundary value satisfies

$$\lim_{\epsilon \downarrow 0} \frac{1}{2\pi i}(R(\lambda + i\epsilon) - R(\lambda - i\epsilon)) = \frac{dE}{d\lambda}(\lambda), \tag{10}$$

where $E(\lambda)$ is the spectral measure for H. The spectral measure is related with eigenfunctions of H through

$$\frac{dE}{d\lambda}(\lambda) = \mathcal{F}(\lambda)^*\mathcal{F}(\lambda), \tag{11}$$

where $\mathcal{F}(\lambda)^*$ is the Fourier transform generated by eigenfunctions $\phi(\lambda, \omega, x)$ of H:

$$\mathcal{F}(\lambda)^*\varphi(x) = (2\pi)^{-n/2}(2\lambda)^{(n-2)/4} \int_{S^{n-1}} \phi(\lambda, \omega, x)\varphi(\omega)d\omega \tag{12}$$

for $\varphi \in L^2(S^{n-1})$.

In this sense, the investigation of time-dependent behavior of $\exp(-itH)$ is equivalent to the investigation of the stationary oscillation, i.e., the eigenfunctions $\phi(\lambda, \omega, x)$:

$$H\phi(\lambda, \omega, x) = \lambda\phi(\lambda, \omega, x). \tag{13}$$

Using $E(\lambda)$ we can write by (6)

$$\exp(-itH) = \int_{-\infty}^{\infty} e^{-it\lambda} dE(\lambda). \tag{14}$$

In particular, when restricting to a subset B of the absolute continuous spectrum of H, we have

$$\begin{aligned} \exp(-itH)E(B) &= \int_B e^{-it\lambda}\frac{dE}{d\lambda}(\lambda)d\lambda \\ &= \int_B e^{-it\lambda}\mathcal{F}^*(\lambda)\mathcal{F}(\lambda)d\lambda. \end{aligned} \tag{15}$$

Thus we obtain a representation formula of the local clock $\exp(-itH)$ through the stationary oscillation (eigenfunction) $\phi(\lambda, \omega, x)$ by (12).

In this sense, local time is inevitably cyclic through local clock $\exp(-itH)$, which is a reflection of the fact that the time-dependent approach is equivalent to stationary approach.

To conclude, time is just an auxiliary parameter to express the internal oscillations, namely, stationary eigenfunctions.

References

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